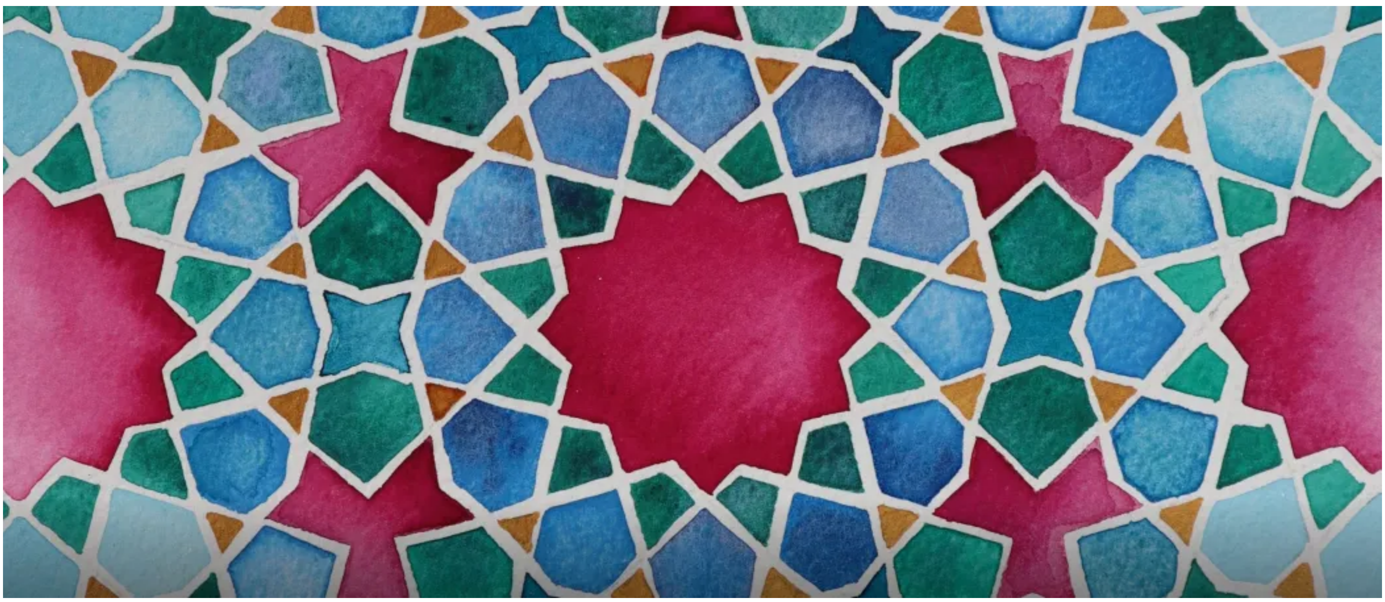
On the cover: Islamic geometry

Explaining the mathematics of tiling, and the cover of Issue 10

Emily Maw 23 October 2019



Samira's

From the exquisite patterns of the Alhambra palace in Spain to a jigsaw puzzle on a rainy day, tessellations (tilings of the plane using shapes with no overlaps or gaps) are everywhere. They are sometimes used for practical reasons: providing durable and water-resistant surfaces, or for efficiencies of space (like hexagons in a honeycomb). And sometimes they are there for aesthetic reasons: tessellations are known to have been used in architecture since at least 4000BC when the Sumerians decorated walls with patterns of clay tiles.

Later, mosques were often decorated with tessellations—alongside calligraphy and arabesque plant forms—to avoid depicting people or animals that could become objects of worship. The repeating patterns are also said to be reminders of the infinite nature of God. Some of the earliest-known geometric forms in Islamic art can be found in the Great Mosque of Kairouan in Tunisia, which was established in the 7th century, but the art-form soon spread across the Islamic world and diversified, developing regional characteristics. The significant intellectual contributions of Islamic mathematicians allowed the patterns to become much more complicated. Islamic designs are constructed on grids that require only a straight edge and compass to draw, building up from lines and circles that are repeated, overlapped and interlaced to form intricate patterns.

The cover of this issue of Chalkdust features artwork by Samira Mian. The front cover is a pattern found on the shrine of Abdulazim Hasani, and the back cover is from a 'jaali' panel in the 14th century Al Rifa'i Mosque in Cairo. Samira researches historical patterns, and the constructs them by hand using efficient and mathematically-pleasing methods to obtain the correct proportions, before painting them in her own beautiful colour schemes. She used to be a maths teacher, but now runs Islamic geometry workshops—

combining her creativity with her love of ruler and compass constructions. To create some of her patterns for yourself, check out samiramian on YouTube or samira.mian on Instagram. And if you want to have a go at constructing the image on the cover, then instructions will be uploaded to the website in the coming weeks.

Let's tessellate

Tessellations aren't just a beautiful application of ancient mathematics though; they involve lots of modern concepts and there are many problems that are still unsolved. The simplest tessellations are called 'regular tilings'. These use only one regular polygon, and there are three of them: the plane can be tiled using the square, equilateral triangle, or regular hexagon. These tilings are also periodic: repeating patterns with translational symmetry in two directions. If you allow combinations of any shapes, periodic tilings can be categorised based on their symmetries into 17 different 'wallpaper groups'. It is claimed that all 17 can be found in the Alhambra!

Can the plane be tiled by repeating the same irregular shape? Conway came up with a sufficient criterion for deciding if a given shape tiles the plane, but no general rule has yet been found; although we now have a good understanding in the case of convex shapes (those where all interior angles are less than 180°). All triangles tile the plane (two stick together to make a parallelogram), and so does any hexagon in which each pair of opposite sides is parallel. In fact so do all quadrilaterals: two can be stuck together to make a hexagon. There are no tilings for convex n-gons when n is bigger than 6. But what about pentagons?

Regular pentagons can't tile the plane because their internal angle (108°) doesn't divide 360 (although they do tile other surfaces such as the sphere and the hyperbolic plane). In fact, it's not known whether there is any other shape with order-5 rotational symmetry which tiles the plane. But it *is* known that there are fifteen types of irregular, convex pentagon that do tile the plane. Four of them were discovered in the 1970s by Marjorie Rice, an amateur mathematician, after reading an article on tessellations by Martin Gardner in *Scientific American*, and the most recent one was only discovered in 2015 by Casey Mann, Jennifer McLoud-Mann, and David Von Derau. Michaël Rao gave a computer-assisted proof in 2017 that there are no more convex pentagons that work, so we now know all of the irregular, convex polygons that can tile the plane.

Since they can all be made to tile periodically (though some can also be arranged to tile aperiodically) there does not exist a convex polygon that tiles only aperiodically—when a shifted copy of a tiling will never match the original. How about other combinations of shapes? Wang conjectured in 1961 that any set of tiles that tiles the plane can always be arranged to do so periodically. However, in 1966 his student Berger found a set of 20,426 tiles that only tile aperiodically. Robinson reduced it to six in 1971, and in 1974 Roger Penrose discovered an aperiodic set of two tiles. It is still not known if there is a single aperiodic tile (though it would have to be non-convex by our above observation).

You might think that an aperiodic pattern would be entirely without symmetry, but this is not the case! Penrose tilings have no translational symmetry, but they do have symmetries coming from the infinite repetition of any bounded patch of the tiling, and reflective and rotational (order 5 again!) symmetries of those patches. They are also 'self-similar', in that the same patterns occur at bigger and bigger scales (like fractals), meaning they can be created recursively using a process called inflation. Penrose tiling patterns have even been discovered in the structure of 'quasicrystals'—substances with aperiodic order, including many aluminium alloys.

Although these discoveries are relatively recent, there is evidence to suggest that the properties of Penrose tiles were understood in the medieval Islamic world. In 2007 two physicists, Peter Lu and Paul Steinhardt, claimed that patterns on the 13th-century Darb-e Imam shrine in Iran showed a ‘nearly-perfect’ Penrose tiling that could not have been constructed without mathematical knowledge of the tiles’ properties. The same researchers also uncovered a 15th-century scroll that describes how to make self-similar patterns from the tiles, five centuries before their discovery in modern-day science.